International Journal of Theoretical Physics, Vol. 39, No. 6, 2000

Levinson's Theorem for the Nonlocal Interaction in One Dimension

Shi-Hai Dong[1]

Received December 18, 1999

The Levinson theorem for the one-dimensional Schrödinger equation with both local and the nonlocal symmetric potentials is established by the Sturm–Liouville theorem. The critical case where the Schrödinger equation has a finite zero-energy solution is also analyzed. It is shown that the number n_+ (n_-) of bound states with even (odd) parity is related to the phase shift $\eta_+(0)[\eta_-(0)]$ of the scattering states with the same parity at zero momentum as

$$\eta_+(0) = \begin{cases} (n_+ - 1/2)\pi & \text{noncritical case} \\ n_+\pi & \text{critical case} \end{cases}$$

and

$$\eta_-(0) = \begin{cases} n_-\pi & \text{noncritical case} \\ (n_- + 1/2)\pi & \text{critical case} \end{cases}$$

The problems on the positive-energy bound states and the physically redundant state related to the nonlocal interaction are also discussed.

1. INTRODUCTION

The Levinson theorem (Levinson, 1949), important in nonrelativistic quantum scattering theory, established the relation between the total number n_l of bound states with angular momentum ℓ and the phase shift $\delta_l(0)$ of the scattering state at zero momentum for the Schrödinger equation with a spherically symmetric potential $V(r)$ in three dimensions,

[1] Physical and Theoretical Chemistry Laboratory, University of Oxford, Oxford OX1 3QZ, U.K., and Department of Physics, Cardwell Hall, Kansas State University, Manhattan, Kansas 66506; e-mail: dongsh@phys.ksu.edu

0020-7748/00/0600-1529$18.00/0

$$\delta_l(0) - \delta_l(\infty) = \begin{cases} (n_l + 1/2)\pi & \text{when } l = 0 \text{ and a half-bound state occurs} \\ n_l\pi & \text{remaining cases} \end{cases} \quad (1)$$

Newton (1960, 1977a,b, 1982) showed the first line in Eq.(1) for the case where a half-bound state (zero-energy resonance) of the *S* wave occurs.

The Levinson theorem has been proved by several authors with different methods and generalized to different fields (Levinson, 1949; Newton, 1960, 1977a,b, 1982, 1994; Jauch, 1957; Martin, 1958; Ni, 1979; Ma and Ni, 1985; Ma, 1985a–c, 1996; Iwinski *et al.*, 1985, 1986; Rosenberg and Spruch, 1996; Liang and Ma, 1986; Poliatzky, 1993; Blankenbecler and Boyanovsky, 1986; Niemi and Semenoff, 1985; Vidal and LeToureux, 1992; Kiers *et al.*, 1996; de Bianchi, 1994; Martin and de Bianchi, 1996; Portnoi and Galbraith, 1997, 1998; Bollé *et al.*, 1986; Gibson, 1987; Lin, 1997, 1998; Buslaev, 1966; Clark, 1983), including cases with the nonlocal interaction both in three dimensions and in two dimensions (Ma and Dai, 1988; Dong *et al.*, 1998c).

Recently, the direct or implicit study of the one-dimensional Levinson theorem (de Bianchi, 1994; Jackiw and Woo, 1975; Newton, 1980, 1983, 1984; Baton, 1985; Aktosun *et al.*, 1993, 1996, 1998a,b; Nogami and Ross, 1996; Eberly, 1965; van Dijk and Kiers, 1992) has attracted much more attention than that of the two-dimensional one. Consequently, applying the Sturm–Liouville theorem, which is essentially different from other methods used to prove the Levinson theorem, we establish the one-dimensional Levinson theorem for the nonlocal interaction.

This paper is organized as follows. Section 2 establishes the Sturm–Liouville theorem for the nonlocal interaction in one dimension. The corresponding Levinson theorem is set up in Section 3. Some problems on the positive-energy bound states and the physically redundant state associated with the nonlocal interaction are discussed in Sections 4 and 5, respectively.

2. THE STURM–LIOUVILLE THEOREM

Throughout this paper the natural units $\hbar = 1$ and $2\mu = 1$ are employed. Let us consider the one-dimensional Schrödinger equation with a local potential $V(x)$ and a nonlocal potential $U(x, x')$, where both potentials are symmetric:

$$\frac{d^2\,\psi(x)}{dx^2} + [E - V(x)]\psi(x) - \int U(x, x')\psi(x')\,dx' = 0 \quad (2)$$

where E denotes the energy of the particle and

$$V(-x) = V(x), \qquad U(-x, -x') = U(x, x')$$

Introduce a parameter λ for the potentials

$$V(x, \lambda) = \lambda V(x), \qquad U(x, x', \lambda) = \lambda U(x, x') \tag{3}$$

where the potentials $V(x, \lambda)$ and $U(x, x', \lambda)$, when λ increases from zero to one, change from zero to the given potentials $V(x)$ and $U(x, x')$. On introducing the parameter λ, the one-dimensional Schrödinger equation can be modified as

$$\frac{\partial^2}{\partial x^2}\psi(x, \lambda) + [E - V(x, \lambda)]\psi(x, \lambda) = \int U(x, x', \lambda)\psi(x', \lambda)\, dx' \tag{4}$$

Since the potentials are symmetric, the eigenfunction can be combined into those with a definite parity, which satisfy the following boundary conditions at the origin:

$$\begin{aligned} \psi^{(o)}(x, \lambda)\big|_{x=0} &= 0 \qquad \text{for the odd-parity case} \\ \left.\frac{\partial\psi^{(e)}(x, \lambda)}{\partial x}\right|_{x=0} &= 0 \qquad \text{for the even-parity case} \end{aligned} \tag{5}$$

Therefore, we only need discuss the wave function in the range $[0, \infty)$ with the given parities.

It is well known that the mesonic theory of nuclear forces indicates that the interaction between two nucleons is local at a great distance, but becomes nonlocal if the two nucleons come closer. In order to simplify the expression, it is assumed, following Martin (1958), that the nonlocal potential $U(x, x')$ is real, continuous, symmetric, vanishing at large distance, and not too singular at the origin (Chadan, 1958)

$$U(x, x') = \begin{cases} U(x, x') = U(x', x) & \\ U(x, x') = O(x^{-1}) & \text{at} \quad x \sim 0 \\ U(x, x') = 0 & \text{when} \quad x \geq x_0 \end{cases} \tag{6}$$

Generally, the local potential $V(x)$ is real, continuous, and should not be too singular at the origin and at infinity. We assume that $V(x)$ satisfies

$$V(x) = \begin{cases} V(x) = O(x^{-1}) & \text{at} \quad x \sim 0 \\ V(x) = 0 & \text{when} \quad x \geq x_0 \end{cases} \tag{7}$$

These two conditions are necessary for the nice behavior of the wave function at the origin (Levinson, 1949) and the condition of the cutoff potential, respectively. It has been proved that the local potential with a tail at infinity will not change the essence of the proof (Dong and Ma, 1999) if it decays faster than x^{-2} at infinity. Consequently, the integral range in (4) is, in fact, from 0 to x_0, and the equation in the range $[x_0, \infty)$ becomes the case of a free particle.

As defined in our previous work (Dong *et al*, 1998a–c, 1999; Dong and Ma, 2000), we only need one matching condition at x_0, which is the condition for the logarithmic derivative of the wave function (Yang, 1982)

$$A(E, \lambda) \equiv \left\{\frac{1}{\psi(x, \lambda)}\frac{\partial\psi(x, \lambda)}{\partial x}\right\}_{x=x_0-} = \left\{\frac{1}{\psi(x, \lambda)}\frac{\partial\psi(x, \lambda)}{\partial x}\right\}_{x=x_0+} \tag{8}$$

Let us turn to the Sturm–Liouville theorem. Denote by $\overline{\psi}(x, \lambda)$ the solution of Eq. (4) corresponding to the energy $\overline{E}$,

$$\frac{\partial^2}{\partial x^2}\overline{\psi}(x, \lambda) + [\overline{E} - V(x, \lambda)]\overline{\psi}(x, \lambda) = \int U(x, x', \lambda)\overline{\psi}(x', \lambda)\, dx' \tag{9}$$

Multiplying Eqs. (4) and (9) by $\overline{\psi}(x, \lambda)$ and $\psi(x, \lambda)$, respectively, and calculating their difference, we obtain

$$\frac{\partial}{\partial x}\left\{\psi(x, \lambda)\frac{\partial\overline{\psi}(x, \lambda)}{\partial x} - \overline{\psi}(x, \lambda)\frac{\partial\psi(x, \lambda)}{\partial x}\right\} + (\overline{E} - E)\psi(x, \lambda)\overline{\psi}(x, \lambda)$$
$$= \psi(x, \lambda)\int U(x, x', \lambda)\overline{\psi}(x', \lambda)\, dx' - \overline{\psi}(x, \lambda)\int U(x, x', \lambda)\psi(x', \lambda)\, dx' \tag{10}$$

From the boundary condition (5), integrating (10) in the range $[0, x_0]$ and noting the symmetric property of $U(x, x')$, we obtain

$$\frac{1}{\overline{E} - E}\left\{\psi(x, \lambda)\frac{\partial\overline{\psi}(x, \lambda)}{\partial x} - \overline{\psi}(x)\frac{\partial\psi(x, \lambda)}{\partial x}\right\}_{x=x_0-}$$
$$= -\int_0^{x_0}\overline{\psi}(x, \lambda)\psi(x, \lambda)\, dx \tag{11}$$

Taking the limit, we obtain

$$\frac{\partial A(E, \lambda)}{\partial E} = \frac{\partial}{\partial E}\left(\frac{1}{\psi(x, \lambda)}\frac{\partial\psi(x, \lambda)}{\partial x}\right)_{x=x_0-} = -\psi(x_0, \lambda)^{-2}\int_0^{x_0}\psi(x, \lambda)^2\, dx \leq 0 \tag{12}$$

Similarly, from the boundary condition that when $E < 0$ the wave function $\psi(x, \lambda)$ tends to zero at infinity, and when $E = 0$, the derivative of the function is equal to zero at infinity, we have

$$\frac{\partial}{\partial E}\left(\frac{1}{\psi(x,\lambda)}\frac{\partial\psi(x,\lambda)}{\partial x}\right)_{x=x_0+} = \psi(x_0,\lambda)^{-2}\int_{x_0}^{\infty}\psi(x,\lambda)^2\,dx > 0 \tag{13}$$

Therefore, when $E \leq 0$, it is evident that both sides of Eq. (8) are monotonic with respect to the energy E: as the energy increases, the logarithmic derivative of the wave function at x_0- decreases monotonically, but that at x_0+ increases monotonically, which is the essence of the Sturm–Liouville theorem.

3. THE LEVINSON THEOREM

In order to establish the Levinson theorem for the nonlocal interaction, we are now going to solve Eq. (4) in two ranges $[0, x_0]$ and $[x_0, \infty)$, and match two solutions at x_0. According to the condition (5), there exists only one solution near the origin. For example, for the free particle ($\lambda = 0$), the solution of Eq. (4) in the range $[0, x_0]$ is real and can be written as

$$\psi^{(e)}(x, 0) = \begin{cases}\cos(kx) & \text{when} \quad E = k^2 > 0\\ \cosh(\kappa x) & \text{when} \quad E = -\kappa^2 \leq 0\end{cases} \tag{14}$$

for the even-parity case, and

$$\psi^{(o)}(x, 0) = \begin{cases}\sin(kx) & \text{when} \quad E = k^2 > 0\\ \sinh(\kappa x) & \text{when} \quad E = -\kappa^2 \leq 0\end{cases} \tag{15}$$

for the odd-parity case.

In the range $[x_0, \infty)$, $V(x) = U(x, x') = 0$. For $E > 0$, there exist two oscillatory solutions of Eq. (4) whose combination can always satisfy the matching condition (8), so that there exists a continuous spectrum for $E > 0$. Suppose that the phase shifts $\eta_{\pm}(k, \lambda)$ are zero for the free particles ($\lambda = 0$); we have

$$\psi(x, \lambda) = \begin{cases}\cos(kx + \eta_{+}(k, \lambda)) & \text{for the even-parity case}\\ \sin(kx + \eta_{-}(k, \lambda)) & \text{for the odd-parity case}\end{cases} \tag{16}$$

$$\eta_{\pm}(k, 0) = 0 \qquad \text{when} \quad k > 0 \tag{17}$$

Some remarks will be given here. First, the wave function in Eq. (16) seems not to have a definite parity. In fact, the solutions (16) are only suitable in the range $[x_0, \infty)$. The corresponding solutions in the range $(-\infty, -x_0]$ can be obtained according to the parity of the solution. For example, for the odd-parity case, the solution in the range $(-\infty, -x_0]$ is

$$-\sin(k|x| + \eta_{-}(k, \lambda)) = \sin(kx - \eta_{-}(k, \lambda))$$

Second, the solutions (16) for the even-parity case can be rewritten as

$$\sin(kx + \eta_+(k, \lambda) + \pi/2) \tag{18}$$

The $\eta_+(k, \lambda) + \pi/2$ plays the same role in the even-parity case as $\eta_-(k, \lambda)$ in the odd-parity case. Therefore, we only need to establish the Levinson theorem for the odd-parity case, and the Levinson theorem for the even-parity case can be obtained by replacing $\eta_-(k, \lambda)$ with $\eta_+(k, \lambda) + \pi/2$.

In the range $[x_0, \infty)$, the potentials $V(x)$ and $U(x, x')$ are vanishing and do not depend on λ. However, the phase shifts $\eta_\pm(k, \lambda)$ depend on λ through the matching condition (8)

$$\tan \eta_-(k, \lambda) = -\tan(kx_0) \frac{A(E, \lambda) - k \cot(kx_0)}{A(E, \lambda) + k \tan(kx_0)} \tag{19}$$

for the odd-parity case, and a similar formula for the even-parity case can be obtained by replacing $\eta_-(k, \lambda)$ with $\eta_+(k, \lambda) + \pi/2$.

The $\eta_-(k, \lambda)$ are determined from Eq. (19) up to a multiple of π due to the period of the tangent function. Similar to our previous work, for simplicity, we can define

$$\eta_\pm(k) \equiv \eta_\pm(k, 1) \tag{20}$$

Since there is only one finite solution at infinity for $E \le 0$, both for the even-parity case and for the odd-parity case

$$\psi(x, \lambda) = \exp(-\kappa x) \qquad \text{when} \quad x_0 \le x < \infty \tag{21}$$

The solution satisfying the matching condition (8) will not always exist for $E \le 0$. Except for $E = 0$, if and only if there exists a solution of energy E satisfying the matching condition (8) does a bound state appear at this energy, which means that there is a discrete spectrum for $E \le 0$. The finite solution for $E = 0$ is a constant one. It decays not fast enough to be square-integrable such that it is not a bound state if the matching condition (8) is satisfied.

In light of the Sturm–Liouville theorem, we now establish the relation between the number of bound states and the logarithmic derivative $A(0, \lambda)$ of the wave function at $x = x_0-$ for zero energy when the potential changes. For $E \le 0$, we obtain the logarithmic derivative at $x = x_0+$ from Eq. (21),

$$\left(\frac{1}{\psi(x, \lambda)} \frac{\partial \psi(x, \lambda)}{\partial x} \right)_{x=x_0+} = \begin{cases} 0 & \text{when} \quad E \sim 0 \\ -\kappa \sim -\infty & \text{when} \quad E \to -\infty \end{cases} \tag{22}$$

On the other hand, when $V(x) = U(x, x') = 0$, the logarithmic derivative at $x = x_0-$ can be calculated from Eqs. (14) and (15) for $E \le 0$,

$$A(E, 0) = \left(\frac{1}{\psi(x, 0)} \frac{\partial \psi(x, 0)}{\partial x} \right)_{x = x_0 -} = \kappa \tanh(\kappa x_0)$$

$$= \begin{cases} 0 & \text{when} \quad E \sim 0 \\ \kappa \sim \infty & \text{when} \quad E \to -\infty \end{cases} \tag{23}$$

for the even-parity case, and

$$A(E, 0) = \left(\frac{1}{\psi(x, 0)} \frac{\partial \psi(x, 0)}{\partial x} \right)_{x = x_0 -} = \kappa \coth(\kappa x_0)$$

$$= \begin{cases} x_0^{-1} & \text{when} \quad E \sim 0 \\ \kappa \sim \infty & \text{when} \quad E \to -\infty \end{cases} \tag{24}$$

for the odd-parity case

It is evident from Eqs. (22) and (24) that there is no overlap between two variant ranges of two logarithmic derivatives for the odd-parity case, namely there is no bound state for the free particle in the odd-parity case. However, there is one point of overlap from Eqs. (22) and (23). It means that there is a finite solution at $E = 0$ when $\lambda = 0$ for the even-parity case. It is nothing but a constant solution. This solution is finite, but does not decay fast enough at infinity to be square-integrable. It is not a bound state, and is called a half-bound state, which will be discussed later.

Now, both for the even-parity case and for the odd-parity case, if $A(0, \lambda)$ decreases across the value zero as λ increases, an overlap between the variant ranges of two logarithmic derivatives of two sides of $x = x_0$ appears. From the Sturm–Liouville theorem, the overlap means that there must exist one and only one energy for which the matching condition (8) is satisfied, that is, a bound state appears. From the viewpoint of node theory, when $A(0, \lambda)$ decreases across the value zero, a node for the zero-energy solution of the Schrödinger equation comes inward from the infinity, namely, a scattering state changes to a bound state.

As λ increases again, $A(0, \lambda)$ may decrease to $-\infty$, jumps to ∞, and then decreases again across the value zero, so that another overlap occurs and another bound state appears. Note that when the zero point in the zero-energy solution $\psi(x, \lambda)$ comes to $x = x_0$, $A(0, \lambda)$ goes to infinity. It is not a singularity.

Each time $A(0, \lambda)$ decreases across the value zero, a new overlap between the variant ranges of two logarithmic derivatives appears such that a scattering state changes to a bound state. At the same time, a new node comes inward from infinity in the zero-energy solution of the Schrödinger equation. Conversely, each time $A(0, \lambda)$ increases across the value zero, an overlap between those two variant ranges disappears so that a bound state changes back to a scattering state, and simultaneously, a node goes outward and disappears in the zero-energy solution. The number $n_{\pm}$ of bound states is equal to the times that $A(0, \lambda)$ decreases across the value zero as λ increases from zero to one, subtracted by the times that $A(0, \lambda)$ increases across the value zero. It is also equal to the number of nodes in the zero-energy solution. In the next section we will show that this number is nothing but the phase shift at zero momentum divided by π, i.e., $\eta_{-}(0)/\pi$ or $\eta_{+}(0)/\pi + 1/2$.

We should pay some attention to the critical case where $A(0, 1) = 0$. A finite zero-energy solution $\psi(x, 1) = c$ in the range $[x_0, \infty)$ will satisfy the matching condition (8) with the zero $A(0, 1)$. Note that when $A(0, 1) = 0$, the wave function at x_0-, $\psi(x_0, 1)$, must be nonvanishing for the nontrivial solution. The constant c is nothing but the nonvanishing value $\psi(x_0, 1)$. The constant solution is not square-integrable so that it is not a bound state, and is called a half-bound state. As λ increases from a number near and smaller than one and finally reaches one, if $A(0, \lambda)$ decreases and finally reaches the value zero, a scattering state becomes a half-bound state, and no new bound state appears. Conversely, as λ increases to reach one, if $A(0, \lambda)$ increases and finally reaches the value zero, a bound state becomes a half-bound state, i.e., a bound state disappears. This conclusion holds for both the even-parity case and the odd-parity case.

When $\lambda = 0$, the $\eta_{\pm}(k, 0)$ are defined to be zero. As λ increases from zero to one, $\eta_{\pm}(k, 0)$ for $k > 0$ change continuously.

For the odd-parity case, the $\eta_{-}(k, \lambda)$ is calculated by Eq. (19). It is shown that the phase shift $\eta_{\pm}(k, \lambda)$ increases monotonically as the logarithmic derivative $A(E, \lambda)$ decreases:

$$\left.\frac{\partial \eta_{-}(k, \lambda)}{\partial A(E, \lambda)}\right|_{k} = \frac{-k \cos^2 \eta_{-}(k, \lambda)}{\{A \cos(kx) + k \sin(kx)\}^2} \leq 0 \tag{25}$$

The phase shift $\eta_{-}(0, \lambda)$ is the limit of the phase shift $\eta_{-}(k, \lambda)$ as k tends to zero. Therefore, we are only interested in the phase shift $\eta_{-}(k, \lambda)$ at a sufficiently small momentum k, $k \ll 1/x_0$. For the small momentum we obtain from Eq. (19)

$$\tan \eta_{-}(k, \lambda) \sim -(kx_0) \frac{A(0, \lambda) - c^2k^2 - x_0^{-1} + k^2x_0/3}{A(0, \lambda) - c^2k^2 + k^2x_0} \tag{26}$$

where the expansion of $A(E, \lambda)$ for small k is used, and

$$A(E, \lambda) \sim A(0, \lambda) - c^2k^2, \qquad c^2 \geq 0 \tag{27}$$

which is calculated from the Sturm–Liouville theorem (12). In both the numerator and the denominator of Eq. (26) we included the next leading term, which is only useful for the critical cases where the leading terms cancel.

First, it can be seen from Eq. (26) that, except for $A(0, \lambda) = 0$, tan $\eta_-(k, \lambda)$ tends to zero as k goes to zero, i.e., $\eta_-(0, \lambda)$ is always equal to a multiple of π except for $A(0, \lambda) = 0$. When $A(0, \lambda) = 0$, the limit $\eta_-(0, \lambda)$ of the phase shift $\eta_-(k, \lambda)$ is equal to $(n + 1/2)\pi$. It is not important for our discussion except for $A(0, 1) = 0$, which is called the critical case and will be discussed later.

Second, for a sufficiently small k, if $A(E, \lambda)$ decreases as λ increases, $\eta_-(k, \lambda)$ increases monotonically. Assume that in the variant process $A(E, \lambda)$ may decrease through the value zero, but does not stop at this value. As $A(E, \lambda)$ decreases, each time tan $\eta_-(k, \lambda)$ for sufficiently small k changes sign from positive to negative, $\eta_-(0, \lambda)$ jumps by π. However, each time tan $\eta_-(k, \lambda)$ changes sign from negative to positive, $\eta_-(0, \lambda)$ remains invariant. Conversely, if $A(E, \lambda)$ increases as λ increases, $\eta_-(k, \lambda)$ decreases monotonically. As $A(E, \lambda)$ increases, each time tan $\eta_-(k, \lambda)$ changes sign from negative to positive, $\eta_-(0, \lambda)$ jumps by $-\pi$, and each time tan $\eta_-(k, \lambda)$ changes sign from positive to negative, $\eta_-(0, \lambda)$ remains invariant.

Third, as λ increases from zero to one, $V(x, \lambda)$ changes from zero to the given potential $V(x)$ continuously. Each time $A(0, \lambda)$ decreases from near and larger than the value zero to smaller than that value, the denominator in Eq. (26) changes sign from positive to negative and the remaining factor remains positive, such that $\eta_-(0, \lambda)$ jumps by π. Conversely, each time $A(0, \lambda)$ increases across the value zero, $\eta_-(0, \lambda)$ jumps by $-\pi$. Each time $A(0, \lambda)$ decreases from near and larger than the value x_0^{-1} to smaller than that value, the numerator in Eq. (26) changes sign from positive to negative, but the remaining factor remains negative, such that $\eta_-(0, \lambda)$ does not jump. Conversely, each time $A(0, \lambda)$ increases across the value x_0^{-1}, $\eta_-(0, \lambda)$ does not jump either.

Therefore, the $\eta_-(0)/\pi$ is just equal to the times $A(0, \lambda)$ decreases across the value zero as λ increases from zero to one, subtracted by the times $A(0, \lambda)$ increases across that value. As discussed in the previous section, we have proved that the difference of the two times is nothing but the number of bound states η_-, i.e., for the noncritical cases, the Levinson theorem for the nonlocal interaction in one dimension for the odd-parity case can written as

$$\eta_-(0) = n_-\pi \tag{28}$$

Fourth, we now discuss the critical case where the logarithmic derivative $A(0, 1)$ ($\lambda = 1$) is equal to zero. In the critical case, the constant solution $\psi(x) = c$ ($c \neq 0$) in the range $[x_0, \infty)$ for zero energy will match $A(0, 1)$ at x_0. In the critical case, it is obvious that there exists a half-bound state both for the even-parity case and for the odd-parity case. A half-bound state is not a bound state, because its wave function is finite, but not square-integrable. As λ increases from a number near and less than one and finally reaches one, if the logarithmic derivative $A(0, \lambda)$ decreases and finally reaches, but does not cross, the value zero, according to the discussion in the previous section, a scattering state becomes a half-bound state when $\lambda = 1$. On the other hand, the denominator in Eq. (26) is proportional to k^2 such that tan $\eta_{-}(k, 1)$ tends to infinity, i.e., $\eta_{-}(0, 1)$ jumps by $\pi/2$. Therefore, for the critical case the Levinson theorem for the nonlocal interaction in one dimension becomes

$$\eta_{-}(0) - \pi/2 = n_{-}\pi \tag{29}$$

Conversely, as λ increases and reaches one, if the logarithmic derivative $A(0, \lambda)$ increases and finally reaches the value zero, a bound state becomes a half-bound state when $\lambda = 1$, and $\eta_{-}(0, 1)$ jumps by $-\pi/2$. In this case, the Levinson theorem (29) still holds.

Finally, for the even-parity case, the only change is to replace $\eta_{-}(0)$ with $\eta_{+}(0) + \pi/2$. Therefore, the Levinson theorem for the one-dimensional Schrödinger equation for the even-parity case can be written as

$$\eta_{+}(0) = \begin{cases} (n_{+} - 1/2)\pi & \text{for the noncritical case} \\ n_{+}\pi & \text{for the critical case} \end{cases} \tag{30}$$

Note that for the free particle in the even-parity case, there is a half-bound state at $E = 0$. It is the critical case where $\eta_{+}(0) = 0$ and $\eta_{+} = 0$. Combining Eqs. (28)–(30), we obtain the Levinson theorem for the nonlocal interaction in one dimension.

4. POSITIVE-ENERGY BOUND STATES

The question of the positive-energy bound state is of fundamental importance in theoretical nuclear physics and the subject has been of interest for a long time (Gourdin and Martin, 1957; Martin, 1958; Beam, 1969; Bolsterli, 1969; Awin, 1991; Husain *et al.*, 1985; Husain and Awain, 1984). It is well known that, in the case with only a local interaction, the wave function and its first derivative would never vanish at the same point except for the origin, so there does not exist a positive-energy bound state. However, in the case with a nonlocal interaction, Martin showed that the solution with an asymptotic form is not unique when the potential satisfies some conditions (Martin, 1958), i.e., there exists a positive-energy bound state with a vanishing asymptotic form. If a small perturbative potential is added such that the nonlocal potential satisfies the conditions, a positive-energy bound state will appear and the phase shift at this energy increases rapidly by almost π. This can be seen explicitly in the examples given by Martin (1958) and Kermode (1976).

It was pointed out by Kermode that the inverse tangent function is not single-valued and it is physically more satisfactory to include a jump by π to the phase shift at the energy E_0, where a positive-energy bound state occurs. Martin (1958) and Chadan (1958) defined the phase shift to be continuous even at E_0 so that an additional π will be included into $\delta(0) - \delta(\infty)$ for each positive-energy bound state. This is their reason for modifying Levinson's theorem by the term $\sigma\pi$, where σ denotes the number of positive-energy bound states. But according to the viewpoint of Kermode, the Levinson theorem need not be modified.

However, the phase shift at zero energy in our convention does not change, no matter which viewpoint is adapted, i.e., no matter whether the phase shift jumps or not at the energy with a positive-energy bound state. Therefore, the Levinson theorem for the nonlocal interaction in one dimension holds for the cases where positive-energy bound states may occur.

5. REDUNDANT STATE

The resonating group model of the scattering of nuclei or other composite systems derives an effective two-body interaction in which a nonlocal potential appears. There are some physically redundant states which describe Pauli-forbidden states for the compound system, and the physical two-body states must be orthogonal to these redundant states (Tamagaki, 1968). In three dimensions, several authors (Saito, 1968, 1969; Okai *et al.*, 1972; Englefield *et al.*, 1974) proposed a simple non-local term which guarantees the required orthogonality, and verified that it was a good representation of the interaction. If there is just one redundant state represented by the real normalized wave function $U(x)$, then the one-dimensional Saito equation is

$$\frac{d^2}{dx^2}\psi(x) + [E - V(x)]\psi(x) = U(x)\int_0^\infty U(s)\left(\frac{d^2}{ds^2} - V(s)\right)\psi(s)\, ds$$

$$\int_0^\infty U^2(s)\, ds = 1 \tag{31}$$

and

$$E\int_0^\infty U(x)\psi(x)\, dx = 0 \tag{32}$$

The solution of (31) satisfies the orthogonality constraint except for that of zero energy. Saito's nonlocal potential is separable.

If the Schrödinger equation with only a local potential $V(x)$ has a bound state with a negative $-\mathscr{E} < 0$, the corresponding wave function is denoted by $\psi(x)$,

$$\frac{d^2}{dx^2}\psi(x) - V(x)\psi(x) = \mathscr{E}\psi(x), \qquad \int_0^\infty \psi(x)^2\, dx = 1 \tag{33}$$

It is obvious that $U(x) = \psi(x)$ satisfies (31) with zero energy. Therefore, it is a so-called physically redundant state. As far as the mathematical equation (31) is concerned, the redundant state is one of the bound states with zero energy.

ACKNOWLEDGMENTS

S.H.D. is grateful to Prof. Mark S. Child for an invitation to visit the University of Oxford. This work was supported in part by the Royal Society and in part by Division of Chemical Sciences, Office of Basic Energy Sciences, Office of Energy Research, US Department of Energy.

REFERENCES

T. Aktosun, M. Klaus, and C. van der Mee (1993). *J. Math. Phys.* **34**, 2651.
T. Aktosun, M. Klaus, and C. van der Mee (1996). *J. Math. Phys.* **37**, 5897.
T. Aktosun, M. Klaus, and C. van der Mee (1998a). *J. Math. Phys.* **39**, 4249.
T. Aktosun, M. Klaus, and C. van der Mee (1998b). *J. Math. Phys.* **39**, 1957.
A. M. Awin (1991). *Fortschr. Phys.* **39**, 131.
G. Baton (1985). *J. Phys. A* **18**, 479.
J. E. Beam (1969). *Phys. Lett. B* **30**, 67.
R. Blankenbecler and D. Boyanovsky (1986). *Physica D* **18**, 367.
D. Bollé, F. Gesztesy, C. Danneels, and S. F. J. Wilk (1986). *Phys. Rev. Lett.* **56**, 900.
M. Bolsterli (1969). *Phys. Rev.* **182**, 1095.
V. Buslaev (1966). In *Topics in Mathematical Physics*, Vol. 1 (Plenum Press, New York).
K. Chadan (1958). *Nuovo Cimento* **10**, 892.
C. W. Clark (1983). *Phys. Rev. A* **28**, 83.
M. S. de Bianchi (1994). *J. Math. Phys.* **35**, 2719.
S. H. Dong and Z. Q. Ma (2000). *Int. J. Theor. Phys.* **39**, 463.
S. H. Dong, X. W. Hou, and Z. Q. Ma (1998a). *Phys. Rev. A* **58**, 2160.
S. H. Dong, X. W. Hou, and Z. Q. Ma (1998b). *Phys. Rev. A* **58**, 2790.
S. H. Dong, X. W. Hou, and Z. Q. Ma (1998c). *J. Phys. A* **31**, 7501.
S. H. Dong, X. W. Hou, and Z. Q. Ma (1999). *Phys. Rev. A* **59**, 995.
J. H. Eberly (1965). *Am. J. Phys.* **33**, 771.
M. J. Englefield and H. S. M. Shoukry (1974). *Prog. Theor. Phys.* **52**, 1554.
W. G. Gibson (1987). *Phys. Rev. A* **36**, 564.
M. Gourdin and A. Martin (1957). *Nuovo Cimento* **6**, 757.
E. H. Hauge and J. A. Støvneng (1989). *Rev. Mod. Phys.* **61**, 917.
D. Husain and A. M. Awin (1984). *Lib. J. Phys.* **13**, 69.
D. Husain, Suhrabuddin, and A. M. Awin (1985). *Aust. J. Phys.* **38**, 539.
Z. R. Iwinski, L. Rosenberg, and L. Spruch (1985). *Phys. Rev.* **31**, 1229.
Z. R. Iwinski, L. Rosenberg, and L. Spruch (1986). *Phys. Rev. A* **33**, 946.
R. Jackiw and G. Woo (1975). *Phys. Rev. D* **12**, 1643.
J. M. Jauch (1957). *Helv. Phys. Acta* **30**, 143.

M. W. Kermode (1976). *J. Phys.* **A9**, L47.
K. A. Kiers and W. Van Dijk (1996). *J. Math. Phys.* **37**, 6033.
N. Levinson (1949). *Danske Vidensk. Selsk. K. Mat.-Fys. Medd.* **25**, (9).
Y. G. Liang and Z. Q. Ma (1986). *Phys. Rev. D* **34**, 565.
Q. G. Lin (1997). *Phys. Rev. A* **56**, 1938.
Q. G. Lin (1998). *Phys. Rev. A* **57**, 3478.
Z. Q. Ma (1985a). *J. Math. Phys.* **26**, 1995.
Z. Q. Ma (1985b). *Phys. Rev. D* **32**, 2203.
Z. Q. Ma (1985c). *Phys. Rev. D* **32**, 2213.
Z. Q. Ma (1996). *Phys. Rev. Lett.* **76**, 3654.
Z. Q. Ma and A. Y. Dai (1988). *J. Phys. A* **21**, 2085.
Z. Q. Ma and G. J. Ni (1985). *Phys. Rev. D* **31**, 1482.
A. Martin (1958). *Nuovo Cimento* **7**, 607.
P. A. Martin and M. S. de Bianchi (1996). *Eur. Phys. Lett.* **34**, 639.
R. G. Newton (1960). *J. Math. Phys.* **1**, 319.
R. G. Newton (1977a). *J. Math. Phys.* **18**, 1348.
R. G. Newton (1977b). *J. Math. Phys.* **18**, 1582.
R. G. Newton (1980). *J. Math. Phys.* **21**, 493.
R. G. Newton (1982). *Scattering Theory of Waves and Particles*, 2nd ed. (Springer-Verlag, New York), and references therein.
R. G. Newton (1983). *J. Math. Phys.* **24**, 2152.
R. G. Newton (1984). *J. Math. Phys.* **25**, 2991.
R. G. Newton (1994). *Helv. Phys. Acta* **67**, 20.
G. J. Ni (1979). *Phys. Energ. Fort. Phys, Nucl.* **3**, 432.
A. J. Niemi and G. W. Semenoff (1985). *Phys. Rev. D* **32**, 471.
Y. Nogami and C. K. Ross (1996). *Am. J. Phys.* **64**, 923.
S. Okai, S. Saito, and R. Tamagak (1972). *Prog. Theor. Phys.* **47**, 484.
N. Poliatzky (1993). *Phys. Rev. Lett.* **70**, 2507.
M. E. Portnoi and I. Galbraith (1997). *Solid State Commun.* **103**, 325.
M. E. Portnoi and I. Galbraith (1998). *Phys. Rev. B* **58**, 3963.
L. Rosenberg and L. Spruch (1996). *Phys. Rev. A* **54**, 4985.
S. Saito (1968). *Prog. Theor. Phys.* **40**, 893.
S. Saito (1969). *Prog. Theor. Phys.* **41**, 705.
R. Tamakagi (1968). *Prog. Theor. Phys. Suppl.* extra number 242.
W. van Dijk and K. A. Kiers (1992). *Am. J. Phys.* **60**, 520.
F. Vidal and J. LeTourneux (1992). *Phys. Rev. C* **45**, 418.
C. N. Yang (1982). In *Monopoles in Quantum Field Theory,* N. S. Craigie, P. Goddard, and W. Nahm, eds.(World Scientific, Singapore), p. 237.